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EXPLICIT BOUNDED-DEGREE UNIQUE-NEIGHBOR CONCENTRATORS

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Here we solve an open problem considered by various researchers by presenting the first explicit constructions of an infinite family $\mathcal{F}$ of bounded-degree 'unique-neighbor' concentrators Γ; i.e., there are strictly positive constants α and ϵ, such that all $\Gamma = (X, Y, E(\Gamma)) \in \mathcal{F}$ satisfy the following properties. The output-set Y has cardinality $\frac{21}{22}$ times that of the input-set X, and for each subset S of X with no more than $\alpha|X|$ vertices, there are at least $\epsilon|S|$ vertices in Y that are adjacent in Γ to exactly one vertex in S. Also, the construction of $\mathcal{F}$ is simple to specify, and each $\Gamma \in \mathcal{F}$ has fewer than $\frac{7|V(\Gamma)|}{2}$ edges. We then modify $\mathcal{F}$ to obtain explicit unique-neighbor concentrators of maximum degree 3.

1. Introduction

We call a bipartite graph $\Gamma = (I, O, E)$, where $|I| = N$, and $|O|$ is less than $(1-c)|I|$ for some absolute constant $c > 0$, a (α, ϵ)-*unique-neighbor concentrator* if, for each subset S of I such that $|S| \le \alpha|I|$, there are at least $\epsilon|S|$ vertices in O that are adjacent to exactly one vertex in S. (Call I the *input-set* of Γ and O the *output-set* of Γ.) Unique-neighbor concentrators are useful in the construction of self-routing networks (see [2], [9]). However, or any strictly positive α and ϵ, the construction of an infinite family of bounded-degree (α, ϵ)-unique neighbor concentrators has been an open question (see [2], [9]), perhaps because the 'second-largest eigenvalue' method, which has been the main tool used in guaranteeing the expansion properties of a graph, does not appear to be strong enough to prove that

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an infinite family of bounded-degree graphs are unique-neighbor concentrators (see [5]). However, here we answer this open question by presenting for strictly positive constants α and ϵ, an explicit infinite family $\mathcal{F}$ of bounded-degree unique-neighbor concentrators, $\Gamma = (X, Y, E(\Gamma))$, where $|X| = \frac{22|Y|}{21}$.

The construction of each $\Gamma = (X, Y, E(\Gamma)) \in \mathcal{F}$ is an appropriately defined, simple graph product of a certain 'small' bipartite graph H with a 'large' graph Λ that has $|X|$ edges, and is in a certain explicit infinite family $\mathcal{H}$. The graph H is a bipartite graph that has 44 vertices on one side, 21 on the other, and a simple algebraic construction, while $\mathcal{H}$ is an infinite family of 44-regular Ramanujan graphs constructed in [6] and [7]. To show that each $\Gamma \in \mathcal{F}$ is indeed a unique-neighbor concentrator, we first present properties of Λ that were established by N. Kahale [5], whose analysis uses the 'second eigenvalue' method. We later present and establish properties of H. Then we show that the combined properties of Λ and H guarantee that Γ is indeed as claimed.

This work here is related to the work presented in [3], where the authors construct infinite families of unbalanced expanders of degree K (K fixed) and linear expansion $(1-\epsilon)K$, where the output side can be any constant fraction smaller than the input. However, we believe the constructions in this paper are of sufficient independent interest. The degrees of the constructions here are much lower, and the specifications are much simpler. Here we use only simple algebraic graph constructions that have been in the literature since the mid 1980's. The very recent extractor constructions that we use in [3] are not used at all here.

We now describe the layout of the rest of this paper. In § 2 we present and analyze $\mathcal{H}$ and H. Then in § 3 we use $\mathcal{H}$ and H to construct $\mathcal{F}$. We then analyze $\mathcal{F}$ using the properties of $\mathcal{H}$ and H presented in § 2, and conclude that $\mathcal{F}$ is an infinite family of unique-neighbor expanders. In § 4 we lower the maximum degree to 3.

We now specify notation. First, for each positive integer q, let $\mathbf{Z}^q$ denote the set of vectors with q coordinates, where each coordinate is an integer. Next, all graphs in this paper are *undirected* and simple. Also, for every integer $q \geq 2$, we define a $q+1$-regular graph Λ to be *Ramanujan* if all but one of the eigenvalues of the adjacency matrix of Λ are no larger than $2\sqrt{q}$.

2. Auxiliary constructions

In this section, we present and analyze $\mathcal{H}$ and H, used in the construction of § 3.

Theorem 2.1. *Let $\mathcal{H}$ be the infinite family of 44-regular Ramanujan graphs. Then there exists strictly positive constants α and ϵ such that, for any $\Lambda = (V, E) \in \mathcal{H}$, and any subset E' of E such that $|E'| \leq \alpha|E|$, there are at least $\epsilon|E'|$ vertices in Λ that are incident to at least 1, but no more than 7, edges in E'.*

Theorem 2.1 is a direct corollary of the following theorem, proved by N. Kahale [5].

Theorem 2.2 ([4], [5]). *Let $\Lambda = (V, E)$ be a k-regular Ramanujan graph. Then for all $\epsilon > 0$, and for any nonempty subset X of size at most $k^{-1/\epsilon}|V|$, the average degree d of $\Lambda[X]$ satisfies*

$$d \leq (1 + \sqrt{k-1})(1 + O(\epsilon)). \tag{1}$$

The next graph H that we present is a modified graph from that constructed in [1].

Proposition 2.3. *Let $H = (W, T, E_H)$ be the bipartite graph with parts $W = \{w_0, \ldots, w_{43}\}$ and $T = \{t_0, \ldots, t_{20}\}$, where $w_{\iota'}$ is adjacent to t_ι in H if and only if either (I) or (II) holds.*

(I) $\iota' \leq 24$, and either (a) $\iota = 20$, or (b) $\lfloor \frac{\iota'}{5} \rfloor \lfloor \frac{\iota}{5} \rfloor \equiv \iota - \iota' \pmod 5$.

(II) $\iota' \geq 25$, and $\iota' - 25 = \iota$.

Then, if U is any nonempty subset of W that has no more than 7 vertices, there is at least one vertex in T that is adjacent in H to exactly one vertex in U.

We defer the proof of Proposition 2.3 to Appendix I.

3. Construction and analysis of $\mathcal{F}$

We construct an infinite family $\mathcal{F}$ of unique-neighbor concentrators in the following fashion. Let $\mathcal{H}$ be an infinite family of 44-regular Ramanujan graphs explicitly constructed in [6,7], and let $\Lambda = (V, E)$ be a graph in $\mathcal{H}$. Let H, W, and T be as in Proposition 2.3, where W is designated the input-set of H, and T the output-set of H. From Λ and H we construct a bipartite graph Γ where one side has $|E|$ vertices. Then we prove Theorem 3.1, which claims that Γ is indeed a bounded-degree unique-neighbor concentrator, thereby constructing $\mathcal{F}$.

First, let $X = \{x_\gamma | \gamma \in E\}$ be a set of $N = |E|$ vertices, and for each $\nu \in V$, let X_ν be the subset of X where $X_\nu = \{x_\gamma | E \ni \gamma \ni \nu\}$; since Λ is 44-regular, $|X_\nu|$ is 44 for each $\nu \in V$. Then, let $\{H_\nu | \nu \in V\}$, be a collection of $|V|$ graphs each isomorphic to H such that (1) for each $\nu \in V$, the input-set of H_ν (the set of vertices of H_ν that map to W in the isomorphism to H) is X_ν, and (2) the output-sets of the H_ν's are disjoint from X and from each other.

Then Γ is the graph with vertex-set $\cup_{\nu \in V} V(H_\nu)$, and with edge-set $\cup_{\nu \in V} E(H_\nu)$. The input-set of Γ is X, and the output-set Y of Γ is the (disjoint) union of the output-sets of the H_ν's.

First, since Λ is 44-regular, $|E| = |X| = 22|V|$, whereas $|Y| = 21 \times |V|$ (indeed, 21 vertices in the output-set of each H_ν, and $|V|$ such ν), so indeed $|X| = \frac{22|Y|}{21}$. So we next prove that, for each subset S of X such that $|S|$ is no larger than $\alpha|X|$, there are at least $\epsilon|S|$ vertices in Y that are adjacent in Γ to exactly one vertex in S, where α and ϵ are as in Theorem 2.1. We then prove that Γ has maximum degree 25. Then Theorem 3.1 stated below will follow.

First, let Y_ν denote the output-set of H_ν, for each vertex $\nu \in V(\Lambda)$. We note two things.

(A) Each edge in Γ incident to a vertex in Y_ν is also incident to a vertex in X_ν.

(B) The induced subgraph of Γ on $X_\nu \cup Y_\nu$ is isomorphic to H, where X_ν maps to W, and Y_ν maps to T.

Now, let S' be the set $\{\gamma \in E | x_\gamma \in S\}$. Then S' and S have the same cardinality, and furthermore, for each vertex $\nu \in V$, the number of edges of S' incident to ν is the quantity $|X_\nu \cap S|$. So if we set Q to be the set of vertices $\nu \in V$ such that ν satisfies $1 \le |X_\nu \cap S| \le 7$, then $|Q|$ is at least $\epsilon|S'| = \epsilon|S|$, by Theorem 2.1. Furthermore, by **(B)** and Proposition 2.3, for each $\nu \in Q$, there is at least one vertex $v_\nu \in Y_\nu$ that is adjacent in Γ to exactly one vertex in $X_\nu \cap S$ (because $X_\nu \cap S$ satisfies $1 \le |X_\nu \cap S| \le 7$ for each $\nu \in Q$) and, by **(A)**, v_ν is also adjacent in Γ to exactly one vertex in all of S. Therefore, since the Y_ν are disjoint, there are at least $|Q| \ge \epsilon|S|$ vertices in Y that are adjacent in Γ to exactly one vertex in S (indeed, consider the set $\{v_\nu | \nu \in Q\}$). So for any subset S of X of no more than $\alpha|X|$ vertices, there are at least $\epsilon|S|$ vertices in Y that are adjacent in Γ to exactly one vertex in S.

Thus, we now bound the maximum degree of Γ, and then Theorem 3.1 will follow. From **(A)** and **(B)**, each vertex in X has degree no more than $2 \times 5 = 10$, since each $x_\gamma \in X$ is in exactly two X_ν (namely, ν is one of two endpoints of γ), and each vertex in W has degree at most 5 in H. Similar lines of reasoning can be used to show that each vertex in Y has degree at most 25 in Γ, and that there are no more than $\frac{7|V(\Gamma)|}{2} \le 7|X|$ edges in Γ.

We have proved the following theorem.

Theorem 3.1. *$\Gamma = (X, Y, E(\Gamma))$ is an (α, ϵ)-unique-neighbor concentrator with input set X and output set Y, where α and ϵ are as in Theorem 2.1.* ∎

In fact, we can give an algebraic construction of the $\Gamma \in \mathcal{F}$ as $\mathcal{H}$ is constructed in [6]. We present this construction of Γ by presenting the construction of $\mathcal{H} = \{\Lambda_3, \ldots, \Lambda_l, \Lambda_{l+1}, \ldots\}$, in turn by presenting, for any integer l greater than 3, the construction of Λ_l, which has $|\mathrm{PGL}_2(\mathbf{Z}/17^l\mathbf{Z})|$ vertices. First, let us write the group $\mathrm{PGL}_2(\mathbf{Z}/17^l\mathbf{Z})$ as $\hat{V}_l$. Then $V(\Lambda_l)$ is $\hat{V}_l$, so we next specify $E(\Lambda_l)$. To do this, first let Σ be the set of 44 elements of $\mathbf{Z}^4$ where, an element s of $\mathbf{Z}^4$ is in Σ if and only if the 4 coordinates of s, written as a_1, a_2, a_3, a_4, are such that (1) $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 43$; (2) a_2, a_3, and a_4 are odd and a_2 is positive. Then let ε_l be a square root of -1 in the ring $\mathbf{Z}/17^l\mathbf{Z}$ (we can find such a ε_l in $O(l^2)$ time). Next let Σ_l' be the set of 44 matrices in $\hat{V}_l$ such that π is in Σ_l' if and only if there exists an s in Σ such that

$$\pi = \begin{pmatrix} 7a_1 + 7\varepsilon a_2 & 7a_3 + 7\varepsilon a_4 \\ -a_3 + \varepsilon a_4 & a_1 - \varepsilon a_2 \end{pmatrix}, \tag{2}$$

where a_1, a_2, a_3, and a_4 are the 4 coordinates of s.

Then $V(\Lambda_l)$ is the set $\hat{V}_l$, and ν and ν' are adjacent in Λ_l if and only if there is a $\pi \in \Sigma_l$ such that $\nu' = \pi\nu$. So from the algebraic construction of Λ_l follows the algebraic construction of each $\Lambda_l \in \mathcal{H}$, from which follows the algebraic construction of each $\Gamma_l \in \mathcal{F}$. Indeed, first write $\Sigma_l' = \{\pi_0, \ldots, \pi_{43}\}$. Then the parts of Γ_l are X_l and Y_l, where $X_l = E(\Lambda_l)$, and $Y_l = \hat{V}_l \times \{0, \ldots, 20\}$. Then for each $\iota \in \{0, \ldots, 19\}$ and each $\nu \in \hat{V}_l$, the set of vertices of X_l adjacent to each $\langle \nu, \iota \rangle \in Y_l$ is the set $\{\{\nu, \pi_{\iota'}\nu\} | \iota'$ satisfies either
(a) $\iota' = \iota + 24$, or
(b) $\iota' \in \{0, \ldots, 24\}$ and $\iota' - \iota \equiv \lfloor \frac{\iota}{5} \rfloor \lfloor \frac{\iota'}{5} \rfloor \pmod 5$ $\}$.
While the set of vertices in X_l adjacent to $\langle \nu, 20 \rangle \in Y_l$ is the set $\{\{\nu, \pi_{\iota'}\nu\} | \iota' \in \{0, \ldots, 24\}\}$.

In fact, we can algebraically construct a larger family $\tilde{\mathcal{F}}$ of unique-neighbor concentrators, such that, for each N' sufficently large, a graph $\Gamma \in \tilde{\mathcal{F}}$ such that the input set of Γ is no larger than $\frac{22N'}{21}$, and no smaller than N'. This enables us to construct routing networks from our concentrators, using (with slight modification) the techniques in [2]. Fix a q that is any prime that is distinct from 17, and of the form $q = 4a + 1$, where a is a positive integer (such as, say, $q = 5$), and then let l_1 and l_2 be nonnegative integers such that l_2 is at least 4. The construction carries through analogously in the ring in the ring $\mathbf{Z}/17^{l_1}q^{l_2}\mathbf{Z}$ $\mathbf{Z}/17^l\mathbf{Z}$ as it does in the ring $\mathbf{Z}/17^{l_1}q^{l_2}\mathbf{Z}$.

4. Valence-3 unique-neighbor concentrators

In this section, we present an explicit infinite family $\mathcal{F}'$ of unique-neighbor concentrators of maximum degree 3, constructed from the family $\mathcal{F}$ of unique-neighbor concentrators that we constructed in § 3. We do this in the following fashion. Let $\Gamma=(X,Y,E(\Gamma))$ be an arbitrary graph in $\mathcal{F}$ with parts X and Y, and for each vertex $v\in V(\Gamma)$, let τ_v be an arbitrary bijection from the set of positive odd integers less than $2|\mathcal{N}_\Gamma(\{v\})$, to $\mathcal{N}_\Gamma(\{v\})$, where $\mathcal{N}_\Gamma(\{v\})$ is the set of vertices of $V(\Gamma)$ that are adjacent in Γ to v. We construct from Γ and $\{\tau_v\}$ a bipartite graph $\Gamma' = (X',Y',E(\Gamma')) \in \mathcal{F}'$ with parts X' and Y', where $|X'| = 2|E(\Gamma)| - |Y|$, and $|Y'| = 2|E(\Gamma)| - |X|$. We then show that Γ' is unique-neighbor expander by proving Theorem 4.1.

The vertex-set $V(\Gamma')$ of Γ' is $\{\nu_{v,\iota}| \;\; v\in V(\Gamma)$, and $\iota \in [|2\mathcal{N}_\Gamma(\{v\})|-1]\}$. Then $\nu_{v,\iota}$ and $\nu_{\hat{v},\hat{\iota}}$ are adjacent in Γ' if and only if either
(1) $v=\hat{v}$, and $\hat{\iota}\in\{\iota+1,\iota-1\}$, or
(2) ι and $\hat{\iota}$ are both odd, and $\tau_v(\iota)=\hat{v}$, and $\tau_{\hat{v}}(\hat{\iota})=v$.
The input and output sets of Γ' are X' and Y', respectively, where $X' = \{\nu_{v,\iota}\in V(\Gamma')| \;\;$ either (1) $v\in X$, and ι is odd, or (2) $v\in Y$, and ι is even$\}$, and $Y' = V(\Gamma')\setminus X'$.

(For ease of notation, we sometimes write $|\mathcal{N}_\Gamma(\{v\})|$ as $\delta_\Gamma(v)$.) To specify the construction of Γ' in another way, we expand each $v\in V(\Gamma)$ to a path χ_v of $2\delta_\Gamma(v)-1$ vertices, and add an edge between χ_x and χ_y if and only if x and y are adjacent in Γ, such that exactly every other vertex, starting with the endpoints, of χ_v, is incident to exactly one of these edges not in χ_v (and the remaining vertices of χ_v aren't incident to any edges outside of χ_v), for every $v\in V(\Gamma)$. The resulting graph is Γ'.

Then X' is the set of vertices that are either (a) in χ_x, for some $x\in X$, and are incident to an edge outside of χ_x, or (b) in χ_y, for some $y\in Y$, and are not incident to an edge outside of χ_y. The other part Y' is the remaining set of vertices.

It is easy to check that the maximum degree of Γ' is indeed 3. Also, it is not hard to check that X' and Y' are indeed independent. Also, $|X'| = \sum_{x\in X}\delta_\Gamma(x) + \sum_{y\in Y}\delta_\Gamma(y)-1 = 2|E(\Gamma)|-|Y|$, while $|Y'| = 2|E(\Gamma)|-|X|$. We've established in the last section that $|E(\Gamma)|$ is no larger than $7|X|$, and that $|Y|$ is equal to $\frac{21|X|}{22}$, so $|X'|$ is at least $\frac{311|Y'|}{310}$. We state the following theorem.

Theorem 4.1. *Γ' is a $(\frac{\alpha}{9},\frac{\epsilon}{2^{17}})$-unique neighbor concentrator, where α and ϵ are as in Theorem 3.1, and has maximum degree 3.*

We prove Theorem 4.1 in Appendix II.

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5. Appendix I: Proof of Proposition 2.3

Proof. Let W' denote the subset of W consisting of the 25 vertices $w_{\iota'}$; $\iota' \le 24$. Then if $U \cap W'$ contains exactly one vertex w, then w is the only vertex in U adjacent in H to t_{20}. Also, if $U \cap W'$ is empty, then there are $|U|$ vertices in T adjacent to exactly one vertex in U since the induced subgraph of H on $(W \backslash W') \cup T$ is a matching that saturates $W \backslash W'$. So we now assume that $|U \cap W'|$ is at least 2. We then show that there are at least $8 - |U \cap W'|$ vertices in T that are adjacent to exactly one vertex in $U \cap W'$. Since each vertex in $W \setminus W'$ is adjacent to only one vertex in T, Proposition 2.3 will follow.

For all nonnegative integers ι no larger than 24, set $a_\iota = \lfloor \frac{\iota}{5} \rfloor$, and for every graph G and every subset U of $V(G)$, let us write the set of vertices of U adjacent to at least one vertex of G as $\mathcal{N}_G(U)$. Then, for each nonnegative integer ι' no larger than 24, we may think of $\mathcal{N}_H(\{w_{\iota'}\}) \setminus \{t_{20}\}$ as the 'line' that contains precisely the points t_ι; ι satisfies both $a_{\iota'} \in \{0,1,2,3\}$, and $\iota = \iota' + a_\iota a_{\iota'}$, where addition and multiplication done in the 5-element field F_5. Thus, if we prove the following lemma, then we have shown that there are at least $8 - |U \cap W'|$ vertices in T that are adjacent to exactly one vertex in $U \cap W'$, if $|U \cap W'|$ is at least 2, and Proposition 2.3 will then follow.

Lemma 5.1. *Let T' be a set of 20 points of the form (a,b); $a \in F_5 \backslash \{4\}$; $b \in F_5$, and let W' be the set of 25 lines, each line parameterized by the ordered pair (a',b'); $a',b' \in F_5$, where the line parameterized by (a',b') contains points in T' precisely of the form $(a,b): b = b' + aa'$, where all operations are done in the field F_5. Next, for any subset U of W' and any point $p \in T'$, let $\varphi_U(p)$ be the number of lines $l \in U$ such that $p \in l$. Now let U' be any subset of at least 2 lines of W'. Then there are at least $8 - |U'|$ points p in T' such that $\varphi_{U'}(p)$ is exactly 1.*

Proof. For each $j = 0,1,2,3$, let A_j denote the set of 5 points of T' of the form (j,b), where $b \in F_5$. So each line in W' intersects each A_j exactly once. These three facts will be useful.

Fact 1. Let l and l' be distinct lines in W'. Then l and l' intersect at most 1 point in T'.

Fact 2. Suppose that l and l' intersect at A_0, and let σ be the element in F_5 such that l intersects A_1 at $(1,a)$, and l' intersects A_1 at $(1,a+\sigma)$. Then, if l intersects A_2 at $(2,b)$, and A_3 at $(3,c)$, then l' intersects A_2 at $(2,b+2\sigma)$, and A_3 at $(3,c+3\sigma)$.

Fact 3. Let l_1 and l_2 be distinct lines in W' that intersect at a point (j',η'). Next, for each $j \neq j'$, let $(j,\eta_{j,1})$ be the point in A_j that is contained by l_1, and let $(j,\eta_{j,2})$ be the point in A_j that is contained by l_2. Then, for all $j,j'' \neq j'$, the equality $(j''-j')^{-1}(\eta_{j'',2}-\eta_{j'',1}) = (j-j')^{-1}(\eta_{j,2}-\eta_{j,1})$ holds (where all operations are done in F_5).

We next use these facts to prove Lemma 5.1 for when $|U'| = 6$ (which is the hardest). To do this, it suffices to show that there exist two points p such that $\varphi_{U'}(p)$ is exactly 1, and we do this next. Now, write the 6 lines of U' as l_1, l_2, l_3, l_4, l_5, and l_6. We now break the proof of this lemma for when $|U'| = 6$ into two cases.

Case 1. $\varphi_{U'}(p) \leq 2$ for each point $p \in T'$.

Then, because each line in U' covers an even number of points, the number of points p such that p satisfies $\varphi_{U'}(p) = 1$, is even. Therefore, unless we are done, we may assume that $\varphi_{U'}(p)$ is either 2 or 0 for each $p \in T'$, or in other words, each point in T' is covered by exactly 2 lines of U' or none. So let us assume without loss of generality that l_1 and l_2 cover the same point in A_0, similarily, l_3 and l_4 also cover the same point in A_0, and l_5 and l_6 cover the same point in A_0 as well. Now, for each $i \in \{1,\ldots,6\}$, let a_i be the element in F_5 such that line l_i intersects A_i at $(1,a_i)$. Next, let us set $\sigma_{1,2} = a_2 - a_1$; $\sigma_{3,4} = a_4 - a_3$; and $\sigma_{5,6} = a_6 - a_5$. Then, for each $j = 1,2,3$ we can use Fact 2 (and the assumption that each point in T' is covered by exactly 2 lines of U' or none) to glean from the pairs of lines that intersect at A_j an equation of the form

$$\sigma_{1,2} + \varepsilon_{3,j}\sigma_{3,4} + \varepsilon_{5,j}\sigma_{5,6} = 0;\ \varepsilon_{3,j},\ \varepsilon_{5,j} \in \{-1, 1\}. \tag{3}$$

Furthermore, there exist $j_1, j_2 \in \{1,2,3\}$ such that either $\varepsilon_{3,j_1} \neq \varepsilon_{3,j_2}$, or $\varepsilon_{5,j_1} \neq \varepsilon_{5,j_2}$.

Indeed, suppose at A_1, say, l_1 intersects l_5; l_2 intersects l_3, and l_4 intersects l_6. Then from this we can derive the equation $\sigma_{1,2}+\sigma_{3,4}-\sigma_{5,6}=0$ (so $\varepsilon_{3,1}$ is 1 here, and $\varepsilon_{5,1}$ is -1 here). Now suppose that at A_2, say, l_1 intersects l_3; l_2 intersects l_6; and l_5 intersects l_4. Then from this, and Fact 2, we may derive the equation $\sigma_{1,2}-\sigma_{5,6}-\sigma_{3,4}=0$ (so $\varepsilon_{5,2}$ is -1 here, and $\varepsilon_{3,2}$ is -1 here). Now, the assumption that each point in T' is covered by exactly 2 lines of U' or none (and Fact 1), guarantees that both $\varepsilon_{3,j}$, and $\varepsilon_{5,j}$ are in $\{-1,1\}$. Furthermore, if $\varepsilon_{3,j}$ is 1 (is -1) and, $\varepsilon_{5,j}$ is 1 [is -1], then l_2 intersects either l_5 (either l_6), or l_3 [or l_4] at A_j. This and Fact 1 imply that there indeed exist $j_1, j_2 \in \{1,2,3\}$ such that either $\varepsilon_{3,j_1} \neq \varepsilon_{3,j_2}$, or $\varepsilon_{5,j_1} \neq \varepsilon_{5,j_2}$ (otherwise l_2 would intersect the same line at least two distinct points in $A_1 \cup A_2 \cup A_3$).

However, observe that the only way that $\sigma_{1,2}$, $\sigma_{3,4}$, and $\sigma_{5,6}$ could simultaneously solve two distinct equations of the form given by (3) is if at least one of $\sigma_{1,2}$, $\sigma_{3,4}$, $\sigma_{5,6}$ is 0. But that would be impossible by Fact 1 (e.g., if $\sigma_{1,2}$ is 0, then l_1 and l_2 would intersect at both A_0 and A_1). So there is no way that every point in T' can be covered by exactly 2 lines of none of U'. So there must be at least 2 points p such that $\varphi_{U'}(p)$ is exactly 1 (since $\varphi_{U'}(p)$ is no larger than 2 by assumption of Case 1), and so we are done with Case 1.

Case 2. There exists a point p' in T where $\varphi_{U'}(p') \geq 3$.

We break Case 2 up into two cases.

Subcase 2-1. $\varphi_{U'}(p')$ is exactly 3. Then let l_1, l_2, and l_3 be the lines in U' that contain p'. Furthermore, by Fact 1, l_1, l_2, and l_3 cover 3 distinct points in A_j for each $j \in F_5 \setminus \{4, j'\}$, so assume that (A) l_4 and l_5 intersect in $A_{j'}$ (lest there are two points p in $A_{j'}$ such that $\varphi_{U'}(p)$ is exactly 1, and then we are done). We may now assume that (A) both l_4 and l_5 must intersect one of l_1, l_2, l_3 in A_j, for each $j \in F_5 \setminus \{j', 4\}$, otherwise we are done. Now fix any $j \in F_5 \setminus \{j', 4\}$, and for each $i \in \{1, \ldots, 6\}$, write as $(j, \eta_{j,i})$ the point in A_j that intersects l_i. Then, for each $i_1, i_2 \in \{1,2,3\}$, write as σ_{i_1,i_2} the quantity $(\eta_{j,i_2} - \eta_{j,i_1})$, and write as $\sigma_{4,5}$ the quantity $(\eta_{j,5} - \eta_{j,4})$. Note that (*) $\sigma_{i_1,i_2} = -\sigma_{i_2,i_1}$ for each all distinct $i_1, i_2 \in \{1,2,3\}$, and that $\sigma_{i_1,i_2} + \sigma_{i_2,i_3} + \sigma_{i_3,i_1} = 0$ for all distinct $i_1, i_2, i_3 \in \{1,2,3\}$.

So we now use Fact 3 and Fact 1 to finish Subcase 2-1. Facts 3 and 1 and (A) imply that (**) $\sigma_{i_1,i_2}=\sigma_{4,5}$ for at least 3 distinct ordered pairs (i_1,i_2). (Indeed by (A), for each $j\in F_5\setminus\{j',4\}$, there is an i_1, $i_2\in\{1,2,3\}$ such that l_4 intersects l_{i_1} in A_j, and l_5 intersects l_{i_2} in A_j. Fact 3 then implies that $\sigma_{4,5}=\sigma_{i_1,i_2}$. Fact 1 implies that there are $|F_5\setminus\{4,j'\}|=3$ distinct ordered pairs (i_1,i_2), and so (**) follows.) But the only way that all the inequalities of both (*) and (**) can be simultaneously satisfied is if there exists two distinct $i_1,i_2\in\{1,2,3\}$ such that σ_{i_1,i_2} is 0, which is impossible by Fact 1 (i.e., l_{i_1} and l_{i_2} would then intersect at both $A_{j'}$ and A_j. So there must exist a $j''\in F_5\setminus\{4,j'\}$ such that at least one of l_4 and l_5 does not intersect any of l_1,l_2,l_3 in $A_{j''}$, and so at least 4 points of $A_{j''}$ must be covered by a line in U', and thus, we are done with Subcase 2-1.

Subcase 2-2. There is a point $p'=(j',\eta')$ where $\varphi_{U'}(p')\geq 4$. Then for each $j\in F_5\setminus\{j',4\}$, the lines of U' must cover at least 4 points of $A_{j'}$ by Fact 1. This finishes Subcase 2-2, and thus, Case 2. Thus we have shown that there are at least 2 points in $p\in T'$ where $\varphi_{U'}(p)=1$ when $|U'|=6$.

We now show that Lemma 5.1 holds for when $|U'|=7$. Suppose it does not. Then in each A_j, there are at most 3 points covered by the lines in U'; since each line intersects each A_j, there must be at least one point in A_j covered by 3 lines of U', and another covered in A_j by at least 2 lines of U'. So fix a $j'\in F_5\setminus\{4\}$. Then by mimicking the proof in Subcase 2-1 for when $|U'|$ is 6, we can see that there is at least one $j''\in F_5\setminus\{4,j'\}$ such that 4 points of $A_{j''}$ are covered by a line in U', and thus, at least one point p of those 4 points in $A_{j''}$ must satisfy $\varphi_{U'}(p)=1$, so Lemma 5.1 indeed holds for when $|U'|=7$.

The proofs that there are at least $8-|U'|$ points in T' that are covered by exactly one line of U' for the cases when $|U'|=2,3,4$, and 5 follow the lines of reasoning used to prove the cases when $|U'|$ is 6 and 7, but are easier, and will not be given here. Thus, Lemma 5.1 follows. ∎

Proposition 2.3 follows from Lemma 5.1. ∎

6. Appendix II: Proof of Theorem 4.1

Before we prove Theorem 4.1, we introduce notation, and then we make observations that we will use to finish the proof of Theorem 4.1. For each $x\in X$, let X'_x denote the set of $\delta_\Gamma(x)$ vertices $\nu_{x,\iota}$, where ι is odd (so X'_x is a subset of X'), and let Y'_x denote the set of $\delta_\Gamma(x)-1$ vertices $\nu_{x,\iota}$, where ι is even (so Y'_x is a subset of Y'). For each $y\in Y$, let Y'_y denote the set of $\delta_\Gamma(y)$

vertices $\nu_{y,\iota}$, where ι is odd (so Y'_y is a subset of Y'), and let X'_y denote the set of $\delta_\Gamma(y)-1$ vertices $\nu_{y,\iota}$, where ι is even (so X'_y is a subset of X'). Then the X'_v's, where $v\in V(\Gamma)$, partition X', and the Y'_v's partition Y'. We next present the observations that we will use.

Observation 1. For any vertex $x\in X$, every vertex adjacent to a vertex in Y'_x is in X'_x.

Observation 2. Let U be any nonempty subset of X'_y, where y is an arbitrary vertex in Y. Then there are at least 2 vertices in Y'_y that are adjacent in Γ' to exactly one vertex in U.

(Proof: The induced subgraph of Γ' on $X'_y\cup Y'_y$ is the path χ_y, with endpoints $\nu_{y,1}$ and $\nu_{y,2\delta_\Gamma(y)-1}$, and edges $\{\nu_{y,\iota},\nu_{y,\iota+1}\}$ for each positive integer ι less than $2\delta_\Gamma(y)-1$. So if U is any subset of Y'_y that does not contain either endpoint of χ_y, then there are at least 2 vertices in Y'_y that are adjacent in Γ' to exactly one vertex in U. However, X'_y is precisely the set containing every other vertex of χ_y starting with the neighbors of the endpoints, and U is a subset of X'_y. So indeed Observation 2 follows.)

Observation 3. Let U be any nonempty proper subset of X'_x, where x is an arbitrary vertex in X. Then there is at least 1 vertex in Y'_y that is adjacent to exactly one vertex in U.

The proof of Observation 3 is similar in spirit to that of Observation 2.

Observation 4. Let y and $\bar{y}$ be distinct vertices in Y. Then there are no edges in Γ' between X'_y and $Y'_{\bar{y}}$.

Having presented Observations 1–4, we next introduce some notation. Let S' be an arbitrary subset of X' of cardinality no greater than $\frac{\alpha|X'|}{9}$. Then let Q_0 denote the set of vertices $x\in X$ where $S'\cap X'_x = X'_x$, and let S'_0 denote the set of vertices $x'\in S'$ contained in some X'_x where $x\in Q_0$. Let S'_1 denote the vertices of S' in some X'_x, where $x\in X\setminus Q_0$, and let S'_2 denote the set of vertices in some X'_y, where $y\in Y$. So S'_0, S'_1, and S'_2 partition S'. We next prove the following lemma.

Lemma 6.1. *The number l of vertices in Y' adjacent to exactly one vertex in S' is at least* $\max\{\frac{\epsilon|S'_0|}{10}-3|S'_1|,\ \frac{|S'_1|}{9},\ \frac{|S'_2|}{12}-3(|S'_0|+|S'_1|)\}$.

Proof.

Claim 6.1.1. *l is at least* $\frac{\epsilon|S'_0|}{10}-3|S'_1|$.

Proof. We first claim that (a) the number of vertices in Y' adjacent to exactly one vertex in S_0' is at least $\frac{\epsilon|S_0'|}{10}$. Indeed, let $x \in X$ and $y \in Y$ be vertices in Γ. Then by construction of Γ', there is exactly one edge in Γ' from X_v' to Y_y' if and only if v and y are adjacent, otherwise there are no edges. So if we let Q be the set of vertices in Y that are adjacent in Γ to exactly one vertex in Q_0, then, for each $y \in Q$, there is exactly one edge in Γ' between Y_y' and $\cup_{x\in Q_0} X_x' = S_0'$. This implies that, for each $y \in Q$, there is exactly one vertex y_y' in Y_y' that is adjacent to at least one vertex in S_0', and that y_y' is also adjacent to exactly one vertex in S_0'. So to show (a), all we need to show is that $|Q|$ is at least $\frac{\epsilon|S_0'|}{10}$. However, by Theorem 3.1, Q is at least $\epsilon|Q_0|$. Indeed, we first claim that $|Q_0|$ is no larger than $\alpha|X|$. Since $|S'|$ is no larger than $\frac{\alpha|X'|}{9}$, it follows that (i) $|S_0'| \leq 2\alpha|X|$, since $|X'| = 2|E(\Gamma)|$ $-$ $|Y|$ is no larger than $14|X|$, and $S_0' \subseteq S'$. But (ii) $|Q_0| \leq \frac{|S_0'|}{2}$ since each vertex in X has degree at least 2 in Γ (and so $|X_x'| = |S' \cap X_x'| = |S_0' \cap X_x'|$ is at least 2). So by (i) and (ii), $|Q_0|$ is no larger than $\alpha|X|$. So (iii) $|Q|$ is indeed at least $\epsilon|Q_0|$, by Theorem 3.1. So we now use (iii) to show that $|Q|$ is at least $\frac{\epsilon|S_0'|}{10}$. However, $|S_0'|$ is no larger than $10|Q_0|$, because each vertex in X has degree at most 10 in Γ (and so $|X_x'|$ is no larger than 10), so $|Q|$ is indeed at least $\frac{\epsilon|S_0'|}{10}$, and so (a) follows.

Having proved that the number of vertices in Y' adjacent to exactly one vertex in S_0' is at least $\frac{\epsilon|S_0'|}{10}$, we next show that (b) the number of vertices in Y' adjacent to exactly one vertex in $S_0' \cup S_2'$ is at least $\frac{\epsilon|S_0'|}{10}$. However, (b) follows from Observations 2 and 4. Indeed, let y be an arbitrary vertex in Q, where Q is as in the above paragraph. We claim that (b') there is at least one vertex in Y_y' that is adjacent to exactly one vertex in $S_0' \cup S_2'$. Indeed, we've already established in the above paragraph that there is exactly one vertex y_y' in Y_y' that is adjacent to at least one vertex in S_0', and that y_y' is also adjacent to exactly one vertex in S_0'. So if $S_2' \cap X_y'$ is empty, then (b') follows from Observation 4. However, suppose $S_2' \cap Y_y' = U'$ is nonempty. Then by Observation 2, there is at least 2 vertices $y_{1,y}', y_{2,y}'$ in Y_y' adjacent to exactly one vertex in U', so there is at least one vertex in Y_y' adjacent to exactly one vertex in $S_0' \cup U'$ (take the one of $\{y_{1,y}', y_{2,y}'\}$ that is distinct from y_y'). So (b') follows here as well by Observation 4, since no vertex in Y_y' is adjacent to any other vertex in $S_2' \setminus U'$ by Observation 4. So indeed, (b) follows from (b').

Claim 6.1.1 follows from (b), and the fact that the maximum degree of Γ' is 3.

Claim 6.1.2. *l is at least $\frac{|S_2'|}{12} - 3(|S_1'| + |S_0'|)$.*

Proof. Let $|Q_2|$ be the set of vertices $y \in Y$ such that X'_y is nonempty (so S'_2 is the set of vertices $x' \in X'$ such that x' is also in some X'_y for some $y \in Q_2$). By Observation 2, there are at least $2|Q_2|$ vertices in Y' adjacent to exactly one vertex in S'_2. Since each vertex in Y has degree no larger than 25 in Γ, $|X'_y|$ is no larger than 24 for each such $y \in Y$, so $24|Q_2|$ is at least $|S'_2|$. So there are at least $\frac{|S'_2|}{12}$ vertices in Y' that are adjacent to exactly one vertex in S'_2. Since each vertex in Γ' has degree at most 3, Claim 6.1.2 follows.

Claim 6.1.3. *l is at least $\frac{|S'_1|}{9}$.*

Proof. Let Q_1 be the set of vertices x of X where $S'_1 \cap X'_x$ is nonempty. By Observation 3, for each $x \in Q_1$, there is at least one vertex y'_x in Y'_x that is adjacent to exactly one vertex in $S'_1 \cap X'_x$. However, by Observation 1, that vertex y'_x is also adjacent to exactly one vertex in S'. So there are at least $|Q_1|$ vertices in Y' adjacent to exactly one vertex in S'. However, each vertex in X has degree at most 10 in Γ, so by definition of S'_1, there can be at most 9 vertices in $X'_x \cap S'$ for each $x \in Q_1$, so $|Q_1|$ is at least $\frac{|S'_1|}{9}$. So Claim 6.1.3, and thus Lemma 6.1, follows. ∎

For every positive ϵ, and any nonnegative real numbers a_0, a_1, and a_2, observe that the quantity $\max\{\frac{\epsilon a_0}{10} - 3a_1, \frac{a_1}{9}, \frac{a_2}{12} - 3(a_0 + a_1)\}$ is at least $\frac{a}{2^{17}}$, where $a = a_0 + a_1 + a_2$. So from Lemma 6.1, we can conclude that Γ' is a $(\frac{\alpha}{9}, \frac{\epsilon}{2^{17}})$-unique neighbor concentrator, and Theorem 4.1 follows. ∎

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